

IAS Group Meeting:
Review Of GMN Papers
Sept. 18, 2019



Outline

PUT UP BEFORE TALK

1. Introduction
2. $N=2$ $d=4$
3. Wall Crossing
4. Defects & New BPS Degeneracies
5. Hyperkähler Geometry
6. Derivation of KSWCF
7. Important Special Case: Class S'
8. Defects In Class S
9. Spectral Networks
10. W.I.P.
 - a.) Categorification of WCF
 - b.) Number Theory
11. Further Reading

1. Introduction

Juan asked me to give a review of a project I worked on with A. Neitzke and D. Gaiotto here at IAS from 2008-2012. There were 6 papers, 739 pages plus a minor follow-up by Gaiotto, Witten and myself - another 475 pages. No one wants to read all that, but allegedly there are some results. So I'll sketch some of what's there.

Juan didn't want one particular point, but a broad - necessarily superficial - overview. After some back and forth about what he wanted he told me he wouldn't actually be here - so blame him if this is not what you want!

So for these papers some of the keywords would be:

PUT UP BEFORE TALK!

$d=4, N=2$ field theory; BPS Spectrum;
Wall-crossing; hyperkähler geometry;
hyperholomorphic bundles; WKB analysis;
Stokes phenomenon; M5 branes; $G_2(2,0)$;
Class S' theories; Hitchin systems/Higgs bundles;
Cluster coordinates; Fock-Goncharov coordinates;
line & surface defects; interfaces;
Spectral networks; quantum holonomy;
Quantized character varieties; Landau-Ginzburg
models; A/B models;
Categorified wall-crossing; Fukaya-Seidel cat'.

Project starts w/ BPS states
+ W.C. in $d=4, N=2$

2. Quick $d=4, N=2$ review

$$\{Q_\alpha^i, \bar{Q}_{\dot{\beta}j}\} = 2\sigma_{\alpha\dot{\beta}}^\mu P_\mu \delta^i_j$$

$$\{Q_\alpha^i, Q_\beta^j\} = \epsilon_{\alpha\beta} \epsilon^{ij} \bar{Z}$$

[Do I need to explain notation?]

$$\mathcal{H}_{\text{BPS}} := \{ \psi \mid H\psi = |Z| \psi \}$$

Olive+Witten '77 ; Seiberg+Witten '94

$$\mathcal{M}_{\text{vac}} = \mathcal{M}_{\text{Higgs}} \cup \dots \cup \underbrace{\mathcal{M}_{\text{Coulomb}}}_{\text{broken } SU(2)_R}$$

Unbroken abelian gauge symmetry

$$U(1) \text{ v.m. } \begin{matrix} A_\mu \\ \lambda_1^2 & \lambda_2^2 \end{matrix}$$

$$\varphi \longrightarrow \langle \varphi \rangle$$

$U(1)^r \Rightarrow$ r -dim cplx space of vacua

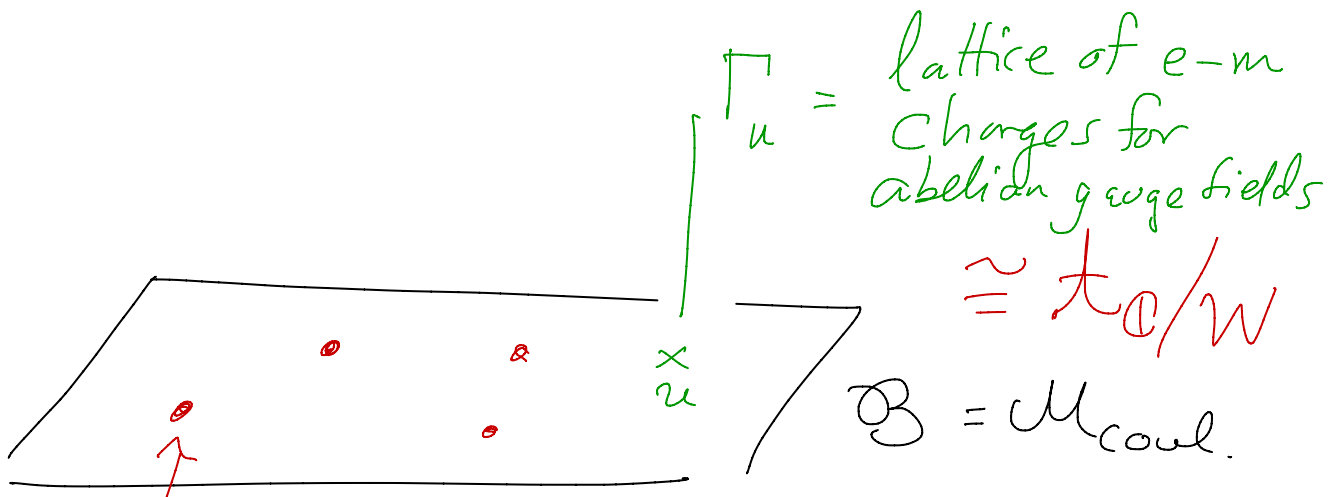
Two Basic Problems

Problem 1: Describe LEET on
Coulomb branch.

Problem 2: Describe BPS spectrum

* SW'94 + 10^4 followup papers solves
Problem 1 for a large class of theories

* GMN solves Problem 2 for "class S"



$\mathcal{B}_{\text{sing}}$: Cplx codimension one: New massless particles

• $\Gamma \rightarrow \mathcal{B}^* = \mathcal{B} - \mathcal{B}_{\text{sing}}$ local system with monodromy

• $\langle \gamma_1, \gamma_2 \rangle \in \mathbb{Z}$ antisymmetric

• $\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma_u} \mathcal{H}_{\gamma, u} \quad \mathbb{Z} \Big|_{\mathcal{H}_{\gamma, u}} = \mathbb{Z}_{\gamma}(u) \updownarrow$
 $\mathbb{Z}_{\gamma_1 + \gamma_2} = \mathbb{Z}_{\gamma_1} + \mathbb{Z}_{\gamma_2}$

$N=2, d=4 \Rightarrow$

Functions \mathbb{Z}_{γ} completely determine LEE1.

Discovery of Seiberg-Witten + followup:

For a large class of $d=4, \mathcal{N}=2$ field theories

\exists hole family (Σ_u, λ_u)
 $\uparrow \quad \uparrow$
R.S. merom 1-form

$$\text{s.t. } \Gamma = H_1(\Sigma_u, \mathbb{Z}) \quad (2)$$

$$Z_\gamma(u) = \oint_\gamma \lambda_u$$

\Rightarrow Solution to Problem 1, BUT

I do not know a general construction of (Σ_u, λ_u) for a general $d=4, \mathcal{N}=2$ field theory.

Problem 2:

$$\mathcal{H}_u = \bigoplus_{\gamma} \mathcal{H}_{\gamma, u}$$

$$\mathcal{H}_{\text{BPS}, u} = \bigoplus_{\gamma} \mathcal{H}_{\text{BPS}, \gamma, u}$$

Give explicit construction of $\mathcal{H}_{\text{BPS}, \gamma, u}$.

3. WALL CROSSING

$\mathcal{H}_{\text{BPS}, \gamma}$ depend on $u \in \mathcal{B}$
They can change discontinuously
for two reasons:

1.) VM + HM BPS reps pair up
to become nonBPS.

2.) Wall-crossing.

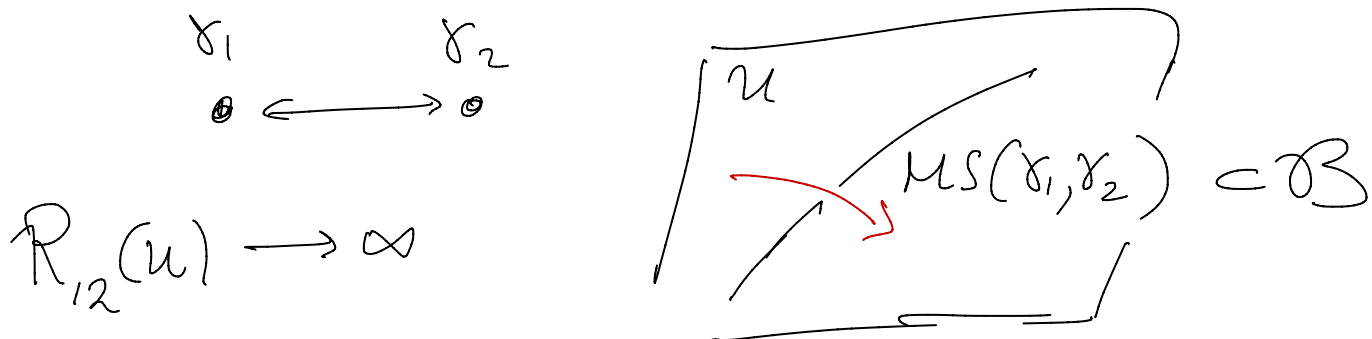
(1) \Rightarrow study index: "protected spin character"

$$\Omega(\gamma; y; u) = \frac{1}{y - y^{-1}} \text{Tr}_{\mathcal{H}_{\text{BPS}, \gamma, u}} (2J_3)^{2J_3} (-1)^{2J_3} (-y)^{2J_3 + 2J_3}$$

Piecewise constant — sometimes can
even be identified with an index of
an elliptic operator — surprise! it
jumps:

BPS particles can form BPS boundstates

CFIV $\frac{1}{2}$ CV 1992; Seiberg-Witten 1994



$$MS(\gamma_1, \gamma_2) := \{u \mid Z_{\gamma_1} \parallel Z_{\gamma_2}\}$$

γ_1, γ_2 prim:

$$\Delta\Omega = \Omega_1 \Omega_2 \chi_j(y)$$

Def
Moore²⁰⁰⁶

$$j = \frac{1}{2}(|\langle \gamma_1, \gamma_2 \rangle| - 1)$$

γ_1, γ_2 nonprimitive: complicated boundstates —

\exists Complicated generalization: (2007)

"Kontsevich - Soibelman WCF" — will derive it in a nice physical way later.

To do that, and get interesting generalizations it is quite useful to broaden our perspective and introduce defects + their BPS states

4. DEFECTS & THEIR BPS STATES

- Line defects : $\text{Cod} = 3$

Preserve $\frac{1}{2} = \frac{4}{8}$ susy \Rightarrow CHOICE OF PHASE \mathcal{S}

e.g. $L = \mathcal{P} \exp \int_{\{\vec{x}=0\} \times \mathbb{R}_t} (\bar{\mathcal{S}}^{-1} \varphi + A + \mathcal{S} \bar{\varphi})$

$$\mathcal{H}_L = \bigoplus_{\gamma_L \in \Gamma + \mathcal{S}_L} \mathcal{H}_{L, \gamma_L}$$

$E \geq -\text{Re}(\bar{\mathcal{S}}^{-1} Z_{\gamma_L})$

\Rightarrow New kind of BPS state:

"framed BPS state" $\psi \in \mathcal{H}_L$ saturating bnd

Framed index:

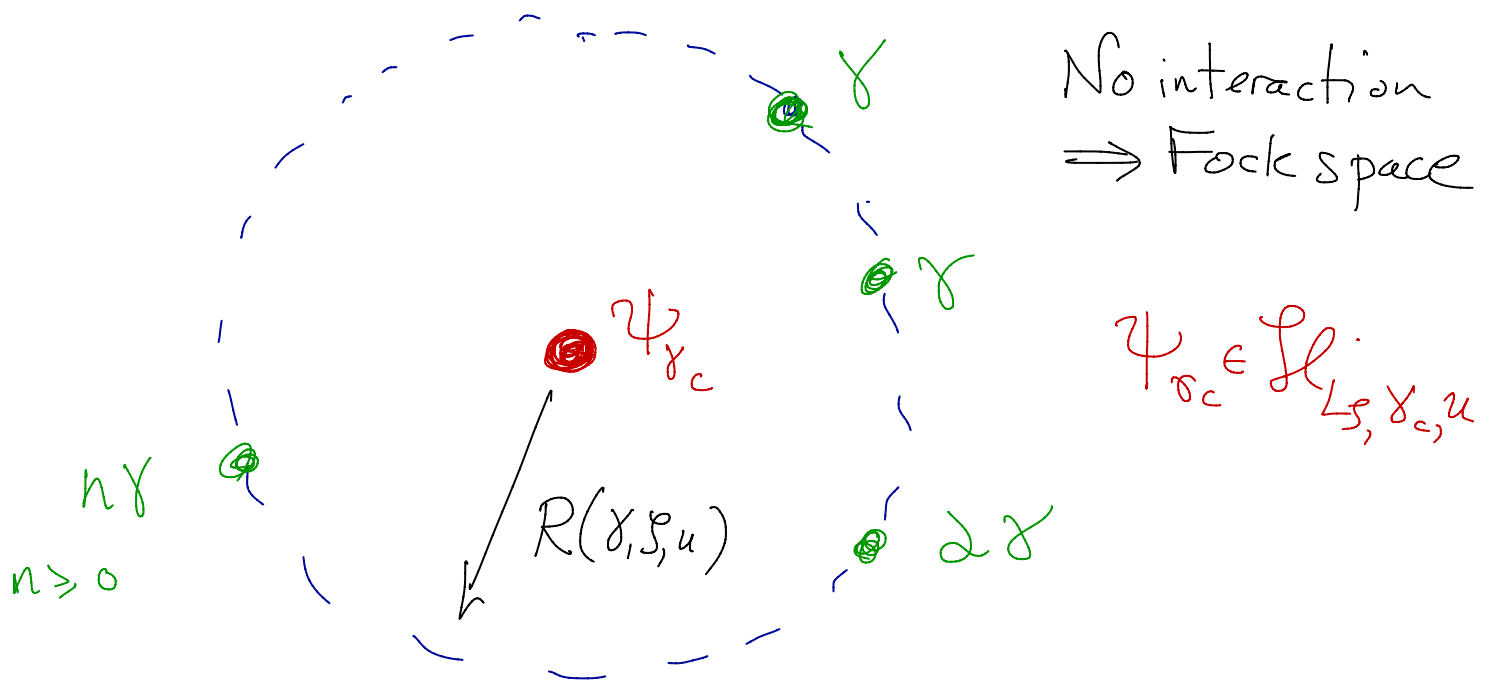
$$\underline{\Omega}(L_{\mathcal{S}, \gamma, y; u) = \text{Tr}_{(\mathcal{H}_L)_{\text{BPS}, \gamma}}^{(-1)^{2J_3}} (-y)^{2J_3 + 2I_3}}$$

Also have wall-crossing - now as functions of (u, \mathcal{S})

$$W(\gamma) := \left\{ (u, \mathcal{S}) \mid \bar{\mathcal{S}}^{-1} Z_\gamma < 0 \right. \\ \left. \text{and } \mathcal{H}_{\text{BPS}, \gamma, u} \neq 0 \right\}$$

"K-wall" or "BPS wall"

near a K-wall there are
half Fock spaces :-



$$R(\gamma, \mathcal{S}, u) = \frac{\langle \gamma, \gamma_c \rangle}{2 \operatorname{Im} (Z_\gamma(u) / \mathcal{S})}$$

For total charge $\gamma_c + N\gamma$

\Rightarrow Fock space appears/disappears across $W(\gamma)$

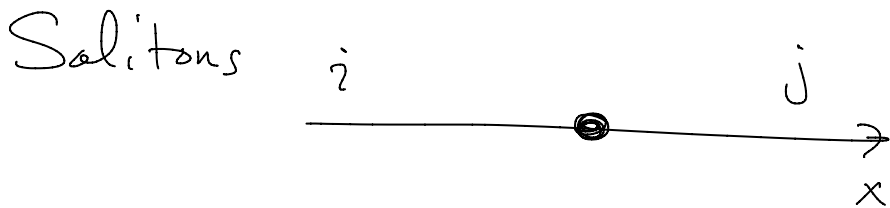
Surface Defects :

Begin with a 2d $N=(2,2)$ QFT

e.g. $LG(X, W)$

Massive
Vacua i, j, k, \dots $dW(\phi_i) = 0$

↑ ↑
Kähler holomorphic potential



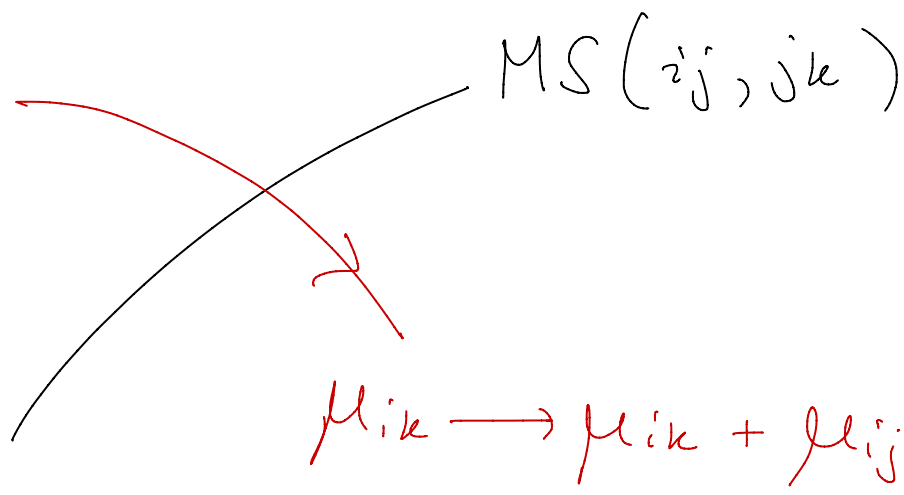
Central charge $Z_{ij} = W(\phi_i) - W(\phi_j)$

index $\mu_{ij} = \text{Tr}_{\mathcal{H}_{ij}} (2J) (-1)^{2J} e^{-\beta H}$

Wall-crossing:

$MS(ij, jk) \subset \text{Space of } W\text{'s}$

$:= \{ W \mid Z_{ij} \parallel Z_{jk} \}$



$$\mu_{ik} \rightarrow \mu_{ik} + \mu_{ij} \mu_{jk}$$

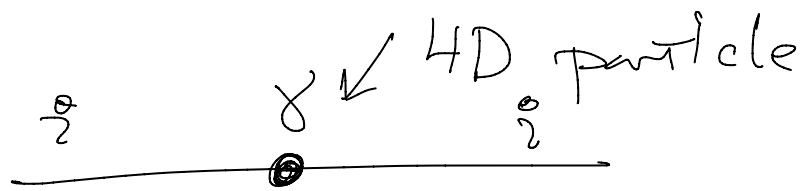
C-V '92

Now suppose 2d $N=(2,2)$ QFT
 has continuous Lie group global G
 symmetry: Embed on $\dim=2$ surface
 in $\mathbb{M}^{1,3}$ and couple to SYM
 with G -gauge symmetry

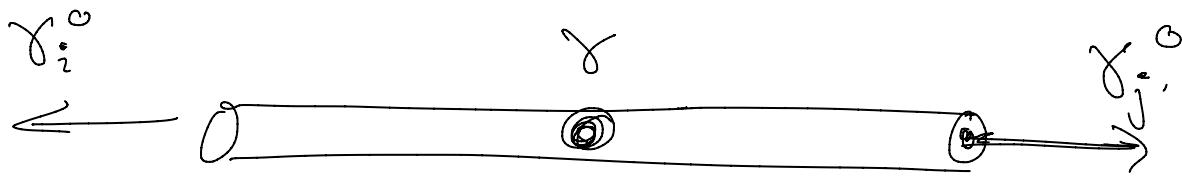
$$\mathbb{M}_{t,x}^{1,1} \times \{(y,z)=(0,0)\} \subset \mathbb{M}^{1,3}$$

"2d-4d system"

\Rightarrow New phenomena



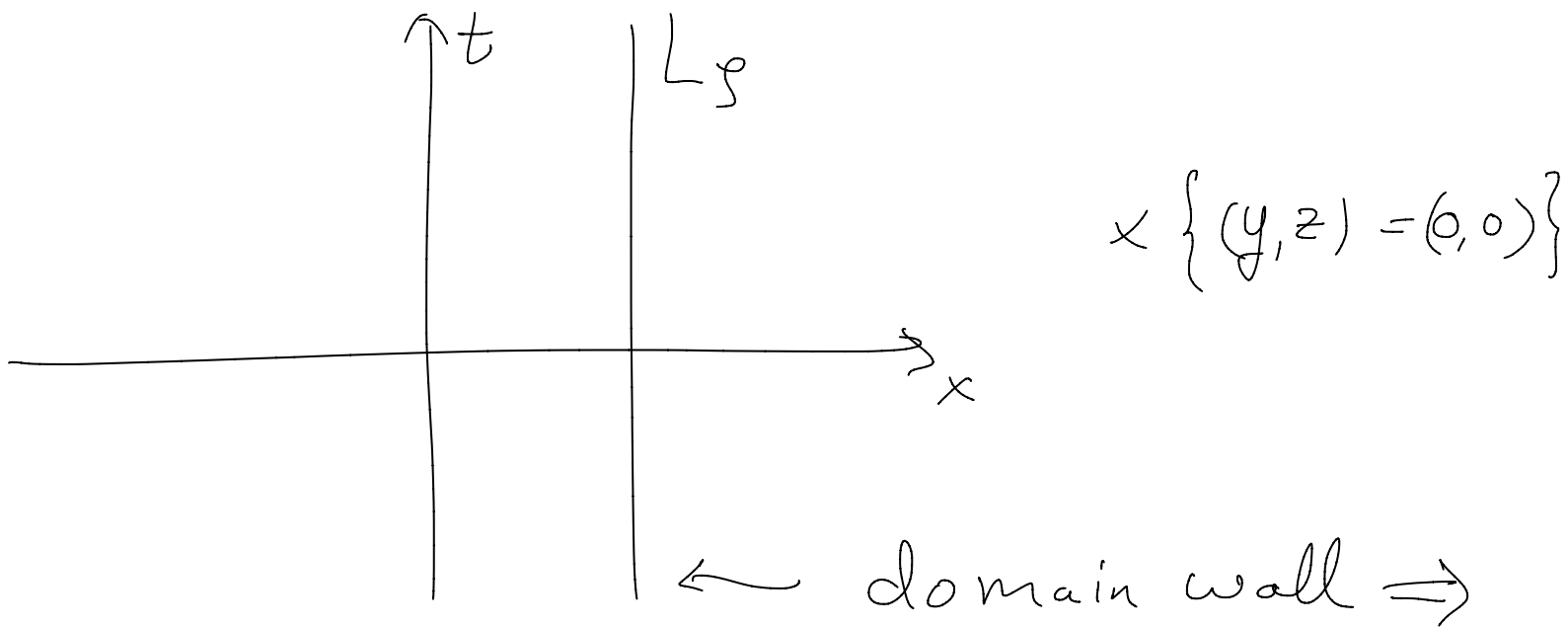
Surface defect functions like a solenoid



\Rightarrow New mixed degeneracies $\omega(\gamma; \gamma_{ij}^0)$

\exists 2d4d wcf for μ_{ij}, ω

- Line defects inside surface defects



\Rightarrow Framed BPS states in the context of 2d QFT: BPS states of $1/2$ -sup interfaces

SCORECARD

5. Hyperkähler Geometry

Compactify : $M^{1,2} \times S^1_R$

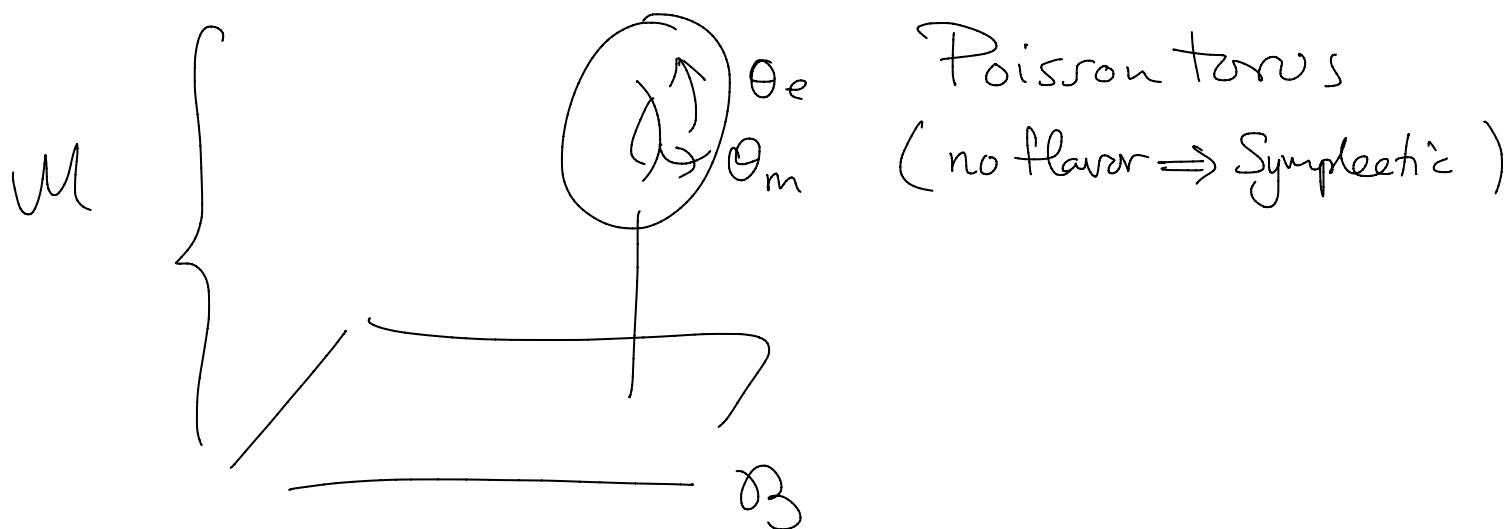
LEFT: σ -model with target \mathcal{M} T_u

$T_u \equiv$ torus \downarrow \downarrow
 $\mathcal{B} \ni u$

$$\Theta_{el} = \lim_{\vec{x} \rightarrow \infty} \oint_{S^1} A_t(\vec{x}, t) dt$$

$d\Theta_m = *F$ $\Theta_m =$ periodic scalar

$$\Theta_m = \lim_{\vec{x} \rightarrow \infty} \Theta_m(\vec{x})$$



$\mathcal{M} =$ Space of 3D vacua has HK metric; S+W '96

Wrap line defects on S^1 :

$\langle L_g \rangle$ is a function on \mathcal{M}

GMN claim:

$$\langle L_g \rangle(u, \theta) = \sum_{\gamma_L \in \Gamma_L} \underbrace{\bar{\Omega}(L_g, \gamma_L, u)}_{@ y = -1} \prod_{\gamma_L} y(u, \theta, S)$$

$y_{\gamma_L}(u, \theta, S)$ locally defined but jump as functions of (u, S)

$\langle L_g \rangle$ has no wall-crossing

$\bar{\Omega}$ has wall-crossing, due to halo Fock spaces,

$\Rightarrow (u, S)$ crosses W_y :

$$y_{\gamma_L} \rightarrow \left(1 + \sigma_r y_r \right) \langle r, \gamma_L \rangle \Omega(r) \prod_{\gamma_L} y_{\gamma_L}$$

$\therefore = K_r^{\Omega(r)} y_{\gamma_L}$

cluster-like tmn: $\pm S \pm mn$.

S.C. limit: $R \rightarrow \infty, u \rightarrow \infty$

$$Y_\gamma(u, \theta, \zeta) \sim \exp\left(\frac{R Z_\gamma}{\zeta} + i\theta \cdot \gamma + R \zeta \bar{Z}_\gamma\right)$$

$\underbrace{\hspace{15em}}_{:= Y_\gamma^{sf}}$

These two conditions essentially determine Y_γ as a solution to a RH problem \Rightarrow

$$\log Y_\gamma(u, \theta, \zeta) = \log Y_\gamma^{sf}(u, \theta, \zeta)$$

$$+ \sum_{\gamma' \in \Gamma} \Omega(\gamma') \int_{-\infty}^{\infty} d\rho' K_{\gamma, \gamma'}(\zeta, \zeta')$$

$$\cdot \log \left[1 + \sigma_\gamma Y_{\gamma'}(u, \theta, \zeta' = e^{i\alpha_{\gamma'} - \rho'}) \right]$$

$$Z_{\gamma'} = e^{i\alpha_{\gamma'}} |Z_{\gamma'}|, \quad \underbrace{\sigma_\gamma \in \{\pm 1\}}_{\text{Q.R. of i.p.}}$$

* The Y_{γ} allow the construction of the HK metric on \mathcal{M} :

$$\begin{aligned}\omega_g^{(2,0)} &= \epsilon^{ij} d \log Y_{\gamma_i} \wedge d \log Y_{\gamma_j} \\ &= \bar{S}^{-1} \omega^{2,0} + \omega^{1,1} + g \omega^{0,2}\end{aligned}$$

$\gamma_i \sim$ basis for $\Gamma \Rightarrow Y_{\gamma_i} \sim$ Darboux coordinates

KSWCF ensures the metric is continuous across $MS(\gamma_1, \gamma_2)$

* The equation is formally identical to Zamolodchikov's TBA - no one knows why.

* \exists similar statements for 2d4d. leads to constructions of hyperbolic vector bundles.

* \exists nice generalization of "Darboux expansion" to $g \neq -1 \Rightarrow$ Noncommutative geometry.

6. DERIVATION OF KSWCF

It is useful to introduce a formal line defect ver:

$$F(L_3) = \sum_{\gamma \in \Gamma_L} \underline{\underline{\Omega}}(L_3, \gamma_L, y, u) X_{\gamma_L}$$

$$X_{\gamma_1} X_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1 + \gamma_2}$$

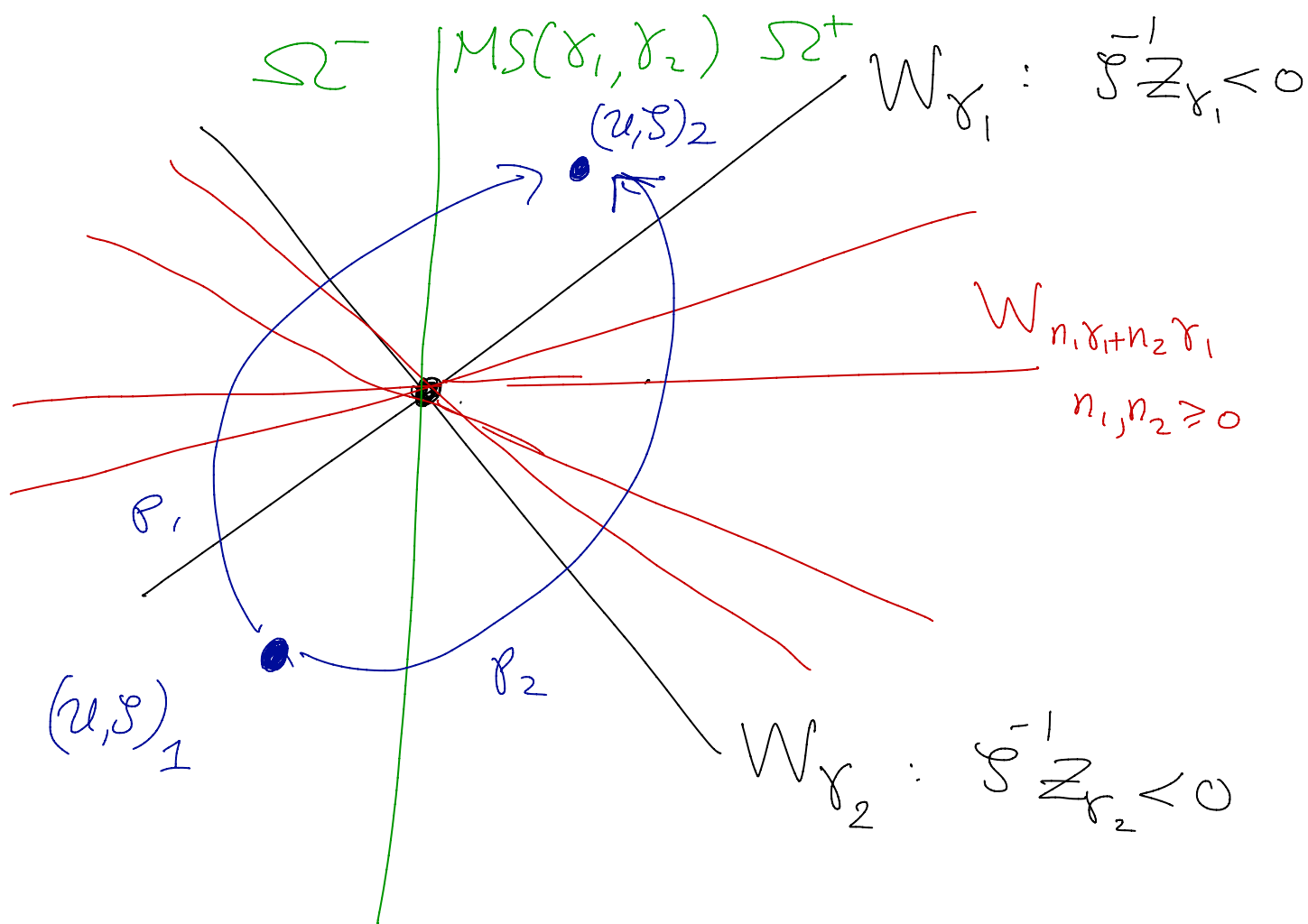
Crossing $W(\gamma)$

$$F(L_3) \rightarrow K(\gamma) F(L_3) K(\gamma)^{-1}$$

Takes into account the creation/ann. of Fock spaces of BPS particles of charge $n\gamma$, $n > 0$.

example: If $\mathcal{H}_{\text{BPS}, \gamma, u}$ only HM's:

$$K(\gamma) = \prod_{k=1}^{\infty} (1 + y^{2k-1} X_{\gamma})^{-\Omega(\gamma)}$$



FRAMED BPS DEG'S ONLY A FUNCTION OF POSITION \Rightarrow

$$\mathcal{P}_1 : F(2) = K(\mathcal{P}_1) F(1) K(\mathcal{P}_1)^{-1}$$

$$\mathcal{P}_2 : F(2) = K(\mathcal{P}_2) F(2) K(\mathcal{P}_2)^{-1}$$

$$\kappa(\rho_1) = \kappa(\gamma_1) \cdots \cdots \kappa(\gamma_2) \left. \vphantom{\kappa(\rho_1)} \right\} \text{using } \Omega^-$$

$$\kappa(\rho_2) = \kappa(\gamma_2) \cdots \cdots \kappa(\gamma_1) \left. \vphantom{\kappa(\rho_2)} \right\} \text{using } \Omega^+$$

Will hold if

$$\overrightarrow{\prod} \kappa(n_1 \gamma_1 + n_2 \gamma_2) = \overleftarrow{\prod} \kappa(n_1 \gamma_1 + n_2 \gamma_2)$$

Using Ω^-
Using Ω^+

This is the (motivic) KSWCF.

(Unipotent property of $\kappa(\gamma)$ rules out potential central term.)

7. Important Special Case: Class S

Here we can say much more.

It involves interesting questions associated to Hitchin systems and flat bundles on Riemann surfaces, hyperbolic geometry of 3-manifolds and more.

"S" is for six because these theories are constructed by starting with a 6d (2,0) theory. [Witten'98; Gaiotto 2009; Gaiotto 2009]

Data: \mathfrak{g} : A-D-E Lie algebra (or sum thereof)
C : Punctured Riemann surface
D : "defect data" at the punctures
explain more below.

Physically: 6d (2,0) theory has cod=2 $\frac{1}{2}$ -BPS defects. We consider theory on

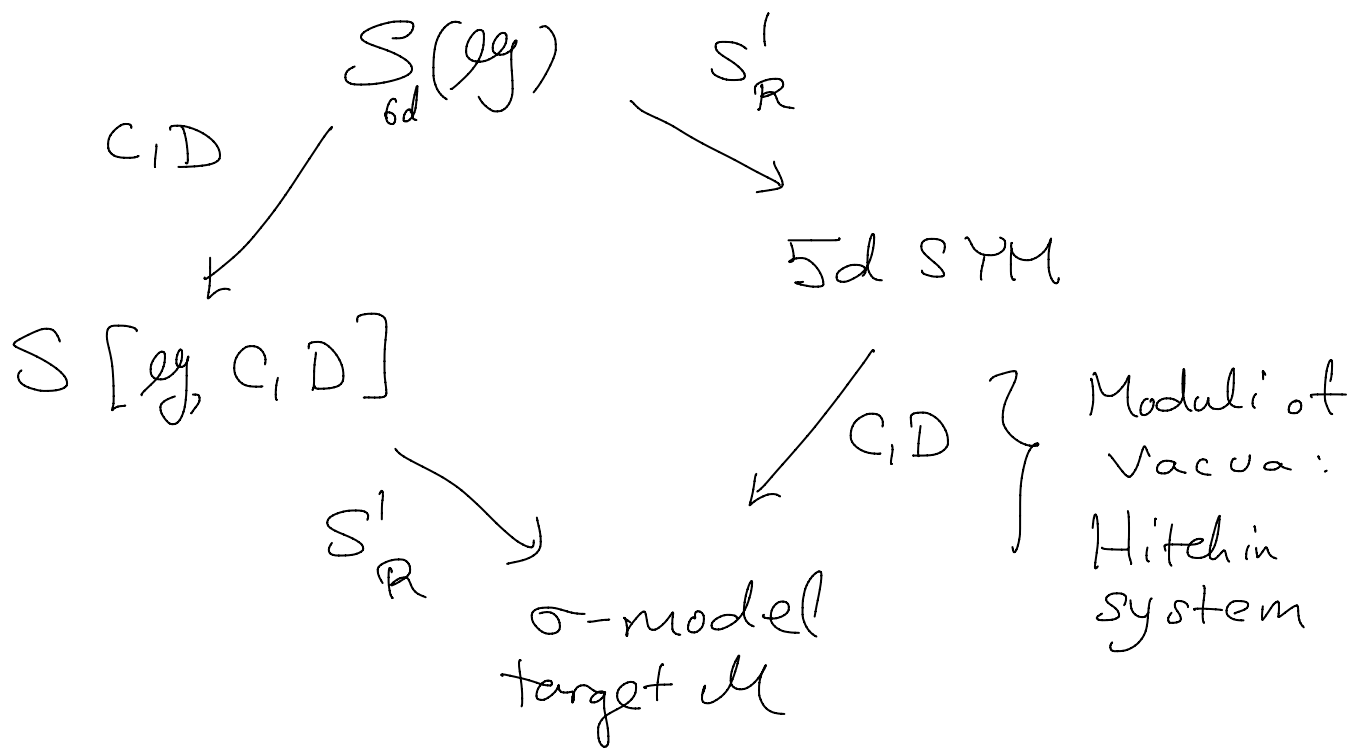
$M^{1,3} \times C$ putting defects @ punctures
filling $M^{1,3}$

Partial topological twist ("class S twist")

- ⇒
- independence of Kähler class of metric on C
 - preservation of 8/16 susy's.

$$\lim_{A(C) \rightarrow 0} 6d [g, C, D] := \underbrace{S[g, C, D]}_{4d \mathcal{N}=2 \text{ theory}}$$

Vacua of this theory are closely related to Hitchin systems:



5d SYM \Rightarrow (with class S twist)

\mathfrak{g} -gauge field A on \mathbb{C}

φ - complex \mathfrak{g} -valued 1-form on \mathbb{C}

$$F + \mathcal{R}^2[\varphi, \bar{\varphi}] = 0$$

$$\bar{\partial}_A \varphi := (\partial_{\bar{z}} \varphi_z + [A_{\bar{z}}, \varphi_z]) dz d\bar{z} = 0$$

Effect of defect at $z=0$:

$$\varphi \sim \frac{r}{z} dz + \text{reg.}$$

$$A \sim \alpha d\theta + \text{reg.}$$

Orbit $(r) \subset \mathfrak{g}_{\mathbb{C}}$ partly characterizes the defect.

Nilpotent orbits $\Rightarrow \exists \mathbb{C}^*$ action \Rightarrow

superconformal

\exists important geodesics with higher order poles.

$\mathcal{M}_{g,n}(\mathbb{C})$ — space of coupling constants

$\Gamma_{g,n,D}$ — duality group

Coulomb branch:

$$\underbrace{\{ \det(\lambda \mathbb{1} - \varphi) = 0 \}}_{\lambda = p dq} \subset T^* \mathbb{C}$$

turns out to be the
Seiberg-Witten curve; $\lambda =$ canonical
SW diff'l

$$\det(\mathbb{1} \lambda - \varphi) = \lambda^n + q_2 \lambda^{n-2} + q_3 \lambda^{n-3} + \dots + q_n$$

$q_k \sim k^{\text{th}}$ diff'l's. holomorphic

with sings @ \mathbb{D}

ex.

$$A_1: \det(\lambda \mathbb{1} - \varphi) = \lambda^2 + q_2$$

8. LINE DEFECTS IN CLASS S

Vacua of $(S[y, c, D] / S'_R)$

= $\mathcal{M}_{\text{Hitchin}}$ with R -dependent hk metric

$\langle L_y \rangle =$ halo function on \mathcal{M} in cplx str. \mathcal{S}

$\mathcal{M}_y =$ moduli of flat $G_{\mathbb{C}}$ -connections with specified "monodromy" @ punctures

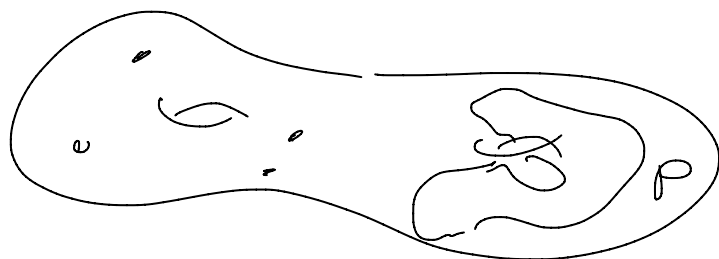
$A = \bar{S}^{-1} \varphi + A + S \bar{\varphi}$ is flat.

From $(2,0)_{6d}$ theory we expect

\exists surface defects labeled by reps \mathcal{R} of \mathfrak{g} . \Rightarrow produce a line defect

associated to closed curve $\rho \subset C$

$L_S(\mathcal{R}, \rho)$



GMN argue:

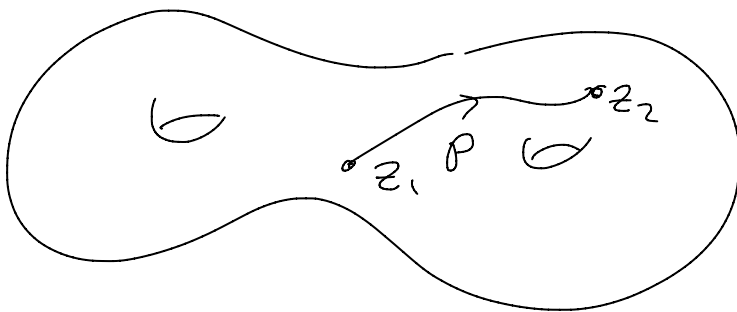
$$\langle L_g(\mathcal{R}, \rho) \rangle = \text{Tr}_{\mathcal{R}} \left(P \exp \int_{\mathcal{R}} A \right)$$

We saw that jumps in the IR/Darboux expansion of line defects lets us derive the 4d BPS spectrum.

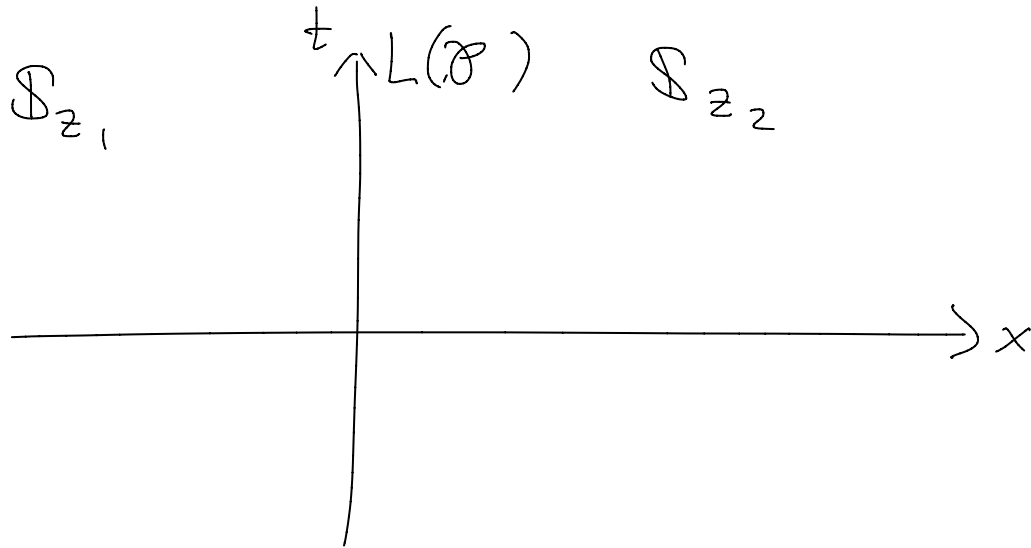
To implement that here it is useful to get at the parallel transport:

$$P \exp \int_{z_1}^{z_2} A$$

Putting 6d (2,0) surface defect @ ZEC defines a surface defect S_z in 4d theory



defines an interface

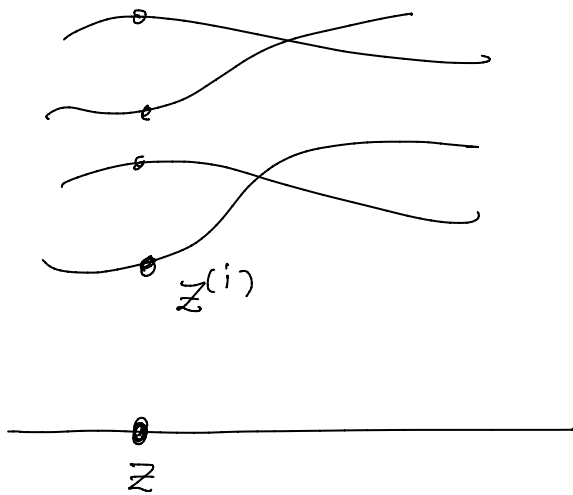


We found a way to construct Pexp SA
Using soliton degeneracies on S_z :
"Spectral networks" -

leads to more math connections
and in this case an explicit
algorithm for computing $\Omega(x)$.

9. SPECTRAL NETWORKS

Identify vacua of \mathcal{S}_z with the points in the fiber $\pi: \Sigma \rightarrow \mathbb{C}$



Solitons have a charge γ_{z_i} associated to homology class of a curve $z^{(i)} \rightsquigarrow z^{(j)}$

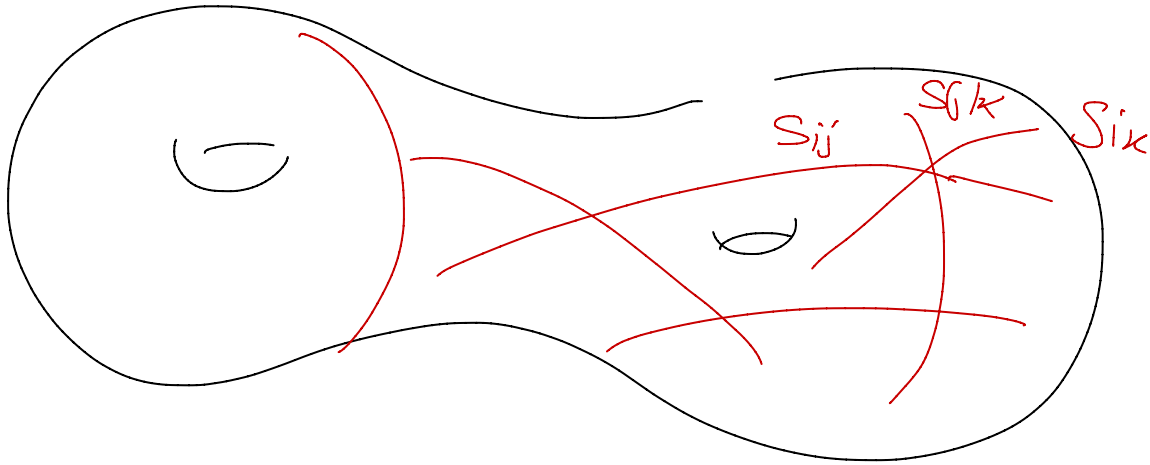
$$\bullet \quad Z_{\gamma_{ij}} = \oint_{\gamma_{ij}} \lambda = W_i - W_j$$

$N=2$ central charge

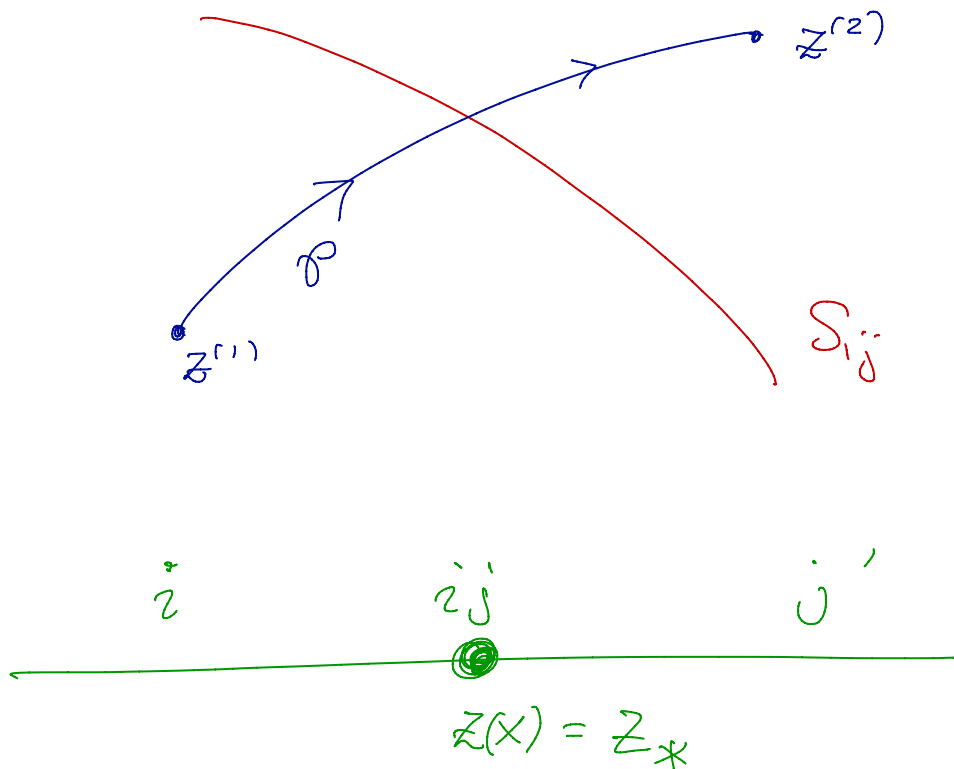
$$\bullet \quad \mu(\gamma_{ij}) = \text{Tr}_{\mathcal{H}(\mathcal{S}_z)} (-1)^F e^{-\beta H}$$

S-WALLS

$$\mathcal{S}(\gamma_{ij}) = \left\{ (z, s) \mid s^{-1} z_{\gamma_{ij}} < 0 \mid \varepsilon \neq 0 \right\}_{\text{BPS } \gamma_{ij} \neq 0}$$



Framed BPS states of interfaces jump when crossing $\mathcal{S}(\gamma_{ij})$ walls:



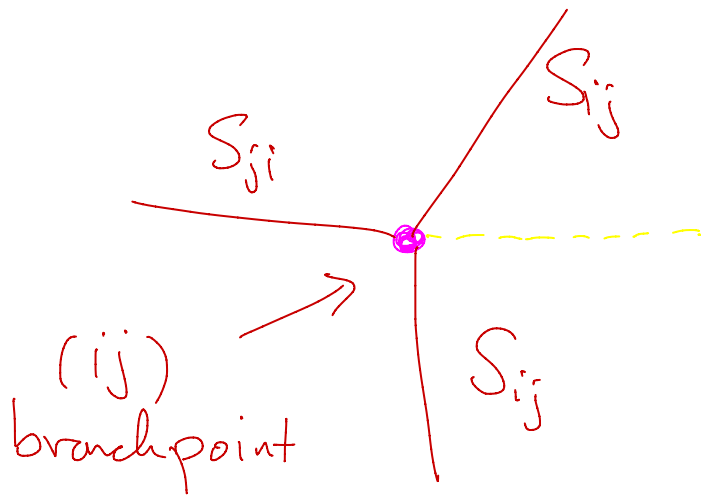
Spectral network is

$$SN(\mathcal{S}) = \left\{ z \mid \exists \text{ soliton with } \bar{\mathcal{S}}' z < 0 \right\} \text{ for theory } \mathcal{S}_z$$

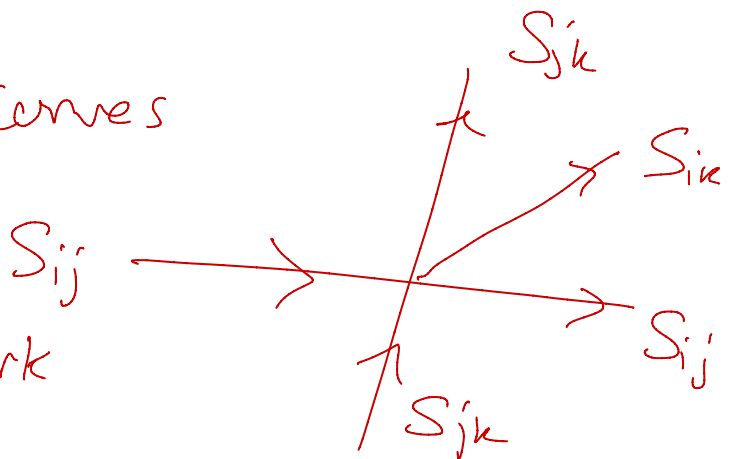
Concretely: Locally, choose a pair of sheets i, j
 Then $\langle \lambda_i - \lambda_j, \partial_t \rangle = -\mathcal{S}$

defines a foliation of G

Look at critical foliation



Grow these curves



\Rightarrow Spectral network $W(\mathcal{S})$

Write $P \exp \int_{z_1}^{z_2} A$ in terms of μ :

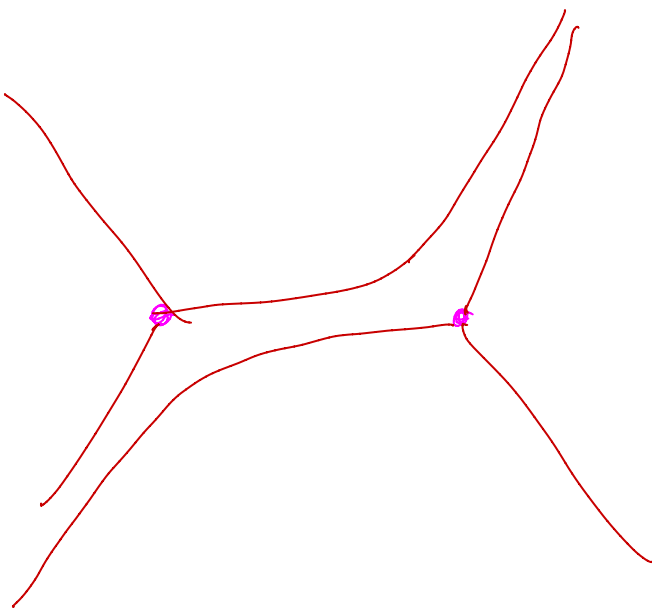
The μ are determined from the SN.

Defines nonabelian connection $\nabla^{M(S), \mu}$

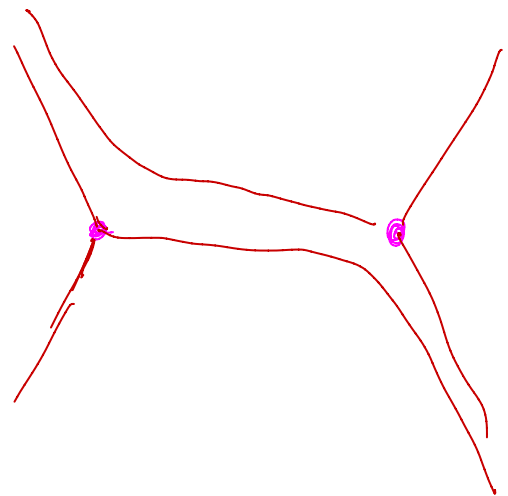
We call this the "nonabelianization map"

Then the change in the SN as

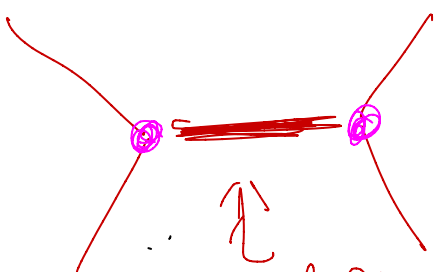
we vary S is due to 4D BPS states:



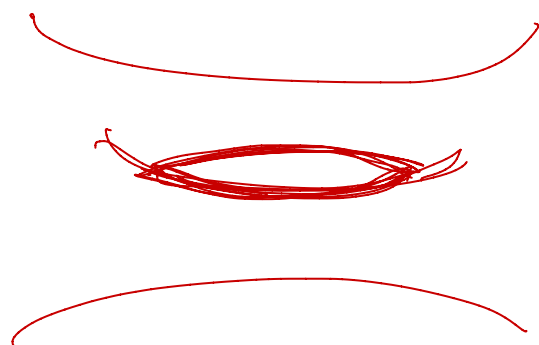
$v < v_c$



$v > v_c$

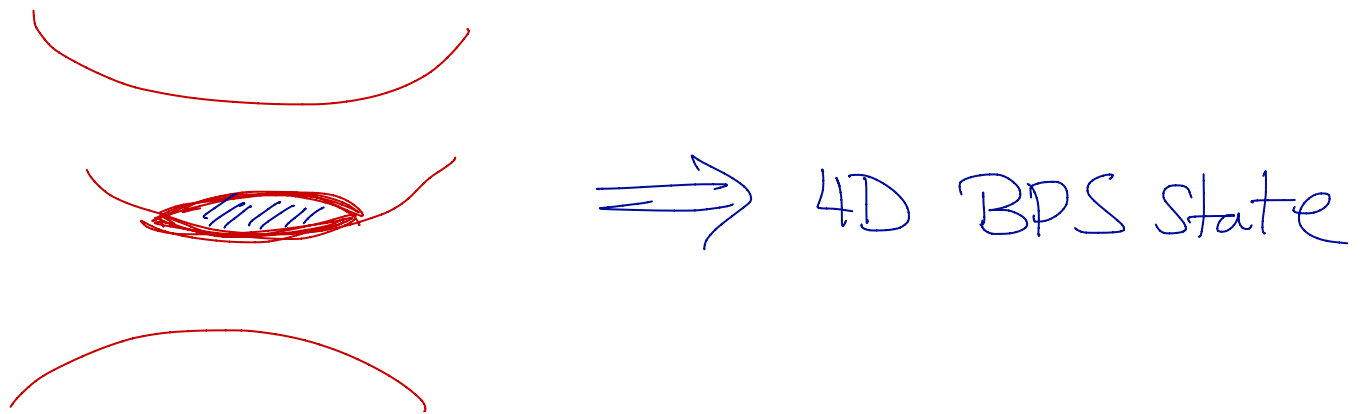


lifts to



CTC*

In M-theory we can fill in
a calibrated disk



The connection $\nabla^{W_{S,u}}$ as defined
by its parallel transport jumps
in a way that depends on $\Omega(x; u)$

The jumps depend on Ω in a
specified way that allows one
to derive Ω .

This is the solution of Problem 2
above for theories of class S .

10. Work In Progress

a.) Categorification

b.) Number Theory

In GMW we wanted to go further — beyond indices — and construct the BPS states themselves and understand how they change under wall-crossing.

It led to an elaborate theory of interfaces between 2 $d=2$ $\mathcal{N}=(2,2)$ theories — and a categorification of the C-V WCF.

Open problem: Generalize to full $2d/4d$ system: Recently we've been making some good progress on that with A. Khan.

Stress here that there is a categorification of S-wall crossing and now we have one for K-wall crossing

11. FURTHER READING

- GMN papers are all on the arXiv
- Short summary of review talk at Intl. Cong. of Maths. 2012: 1211.2331
- Two short reviews by A. Neitzke: 1308.2198 and 1412.7120

Summer school lecture notes:

G. Moore homepage:

#31 GGI Lectures

#35 P;TP Lectures on Wall Crossing

#47 Felix Klein Lectures

#84 Hamburg Higgs Bundles

A. Neitzke: IAS / Park City

PCMI lectures July 2019